II. *On a remarkable Application of* Cotes's *Theorem. By* J. F. W. Herschel, *Esq. Communicated by* W. Herschel, *LL. D. F. R. S.*

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LET a represent the semi-transverse axis of a conic section, ae the eccentricity, and consequently $a(1-e^2)=p$ the semi-parameter.

Let also $\lambda = \frac{e}{1+\sqrt{1-e^2}}$ and $\lambda' = \frac{e^{-1}}{1+\sqrt{1-e^{-2}}}$.

$r^{(1)}$ = the distance between a point in the curve, and the focus, which, for distinction's sake, we shall call the first focus, and the adjacent vertex the first vertex: the others the second.

$r^{(2)}$ = the distance between the same point and the second focus.

R = its distance from the centre.

ρ = its distance from the first vertex.

θ = the angle contained between the $r^{(1)}$, and the *prolongation* of a line joining the first vertex and focus.

ϕ = the angle contained between the R and a line joining the first vertex and centre.

ψ = the angle contained between the ρ and the same line.

$$\tan.\tfrac{1}{2}\varpi = \sqrt{\frac{1-e}{1+e}}.\cot.\tfrac{1}{2}\theta = \frac{1-\lambda}{1+\lambda}.\cot.\tfrac{1}{2}\theta = \sqrt{-1}.\frac{1-\lambda'}{1+\lambda'}.\cot.\tfrac{1}{2}\theta.$$

θ is the angle whose supplement is, in physical astronomy, known by the name of "true anomaly," and ϖ is the corresponding "eccentric anomaly."

The following equations are readily obtained.

$$r^{(1)} = \frac{a(1-e^2)}{1-e \,.\, \cos.\,\theta}; \text{ and } r^{(2)} = 2a - r^{(1)} = a \,.\, \frac{1-2e \,.\, \cos.\,\theta + e^2}{1-e \,.\, \cos.\,\theta}.$$

$$r^{(1)} = a(1-e \,.\, \cos.\,\varpi);\; r^{(2)} = a(1+e \,.\, \cos.\,\varpi)$$

$$R^2 = \frac{a^2(1-e^2)}{1-e^2.(\cos.\varphi)^2}$$

$$\rho = \frac{2a(1-e^2) \,.\, \cos.\,\psi}{1-e^2 \,.\, (\cos.\,\psi)^2} = \frac{2p \,.\, \cos.\,\psi}{1-e^2 \,.\, (\cos.\,\psi)^2}.$$

$$2e = \lambda' + \lambda'^{-1} \text{ and } 2 \,.\, e^{-1} = \lambda + \lambda^{-1}.$$

Hence we deduce the following

$$r^{(1)} = a \,.\, \frac{(1-e^2)(1+\lambda^2)}{1-2\lambda \,.\, \cos.\,\theta+\lambda^2} \quad \ldots\ldots \quad \{1\}$$

$$r^{(2)} = a(1+\lambda^2) \,.\, \frac{1-2e \,.\, \cos.\,\theta+e^2}{1-2\lambda \,.\, \cos.\,\theta+\lambda^2} \quad \ldots\ldots \quad \{2\}$$

$$\frac{r^{(2)}}{r^{(1)}} = \frac{1-2e \,.\, \cos.\,\theta + e^2}{1-e^2} \quad \ldots\ldots \quad \{3\}$$

again $r^{(1)} = a \,.\, \dfrac{1-2\lambda \,.\, \cos.\,\varpi + \lambda^2}{1+\lambda^2} \quad \ldots\ldots \quad \{4\}$

$$r^{(2)} = a \,.\, \frac{1-2\lambda \,.\, \cos.\,(\pi-\varpi) + \lambda^2}{1+\lambda^2} \quad \ldots\ldots \quad \{5\}$$

where $\pi = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.\right\}$.

$$\frac{r^{(1)}}{r^{(2)}} = \frac{1-2\lambda \,.\, \cos.\,\varpi + \lambda^2}{1-2\lambda \,.\, \cos.\,(\pi-\varpi) + \lambda^2} \quad \ldots\ldots \quad \{6\}$$

$$R^2 = \frac{a^2(1-e^2)(1+\lambda^2)^2}{\left\{1-2\lambda \,.\, \cos.\,\varphi+\lambda^2\right\}\left\{1-2\lambda \,.\, \cos.\,(\pi-\varphi) + \lambda^2\right\}} \quad \ldots\ldots \quad \{7\}$$

$$\rho = \frac{2a(1-e^2)(1+\lambda^2)^2 \,.\, \cos.\,\psi}{\left\{1-2\lambda \,.\, \cos.\,\psi+\lambda^2\right\}\left\{1-2\lambda \,.\, \cos.\,(\pi-\psi) + \lambda^2\right\}} \quad \ldots\ldots \quad \{8\}$$

And lastly, since $1-e \,.\, \cos.\,\varpi = \dfrac{1-e^2}{1-e \,.\, \cos.\,\theta}$, we find $\cos.\,\theta =$

$$e^{-1} + \frac{e-e^{-1}}{1-e \,.\, \cos.\,\varpi} = \frac{e(1-e^{-1} \,.\, \cos.\,\varpi)}{1-e \,.\, \cos.\,\varpi}$$

$$\left.\begin{aligned} &= e \,.\, \frac{1+\lambda^2}{1+\lambda'^2} \,.\, \frac{1-2\lambda' \,.\, \cos.\,\varpi+\lambda'^2}{1-2\lambda \,.\, \cos.\,\varpi+\lambda^2} \\ &= e^{-1} \,.\, \left\{\frac{\lambda}{\lambda'}\right\} \,.\, \frac{1-2\lambda' \,.\, \cos.\,\varpi+\lambda'^2}{1-2\lambda \,.\, \cos.\,\varpi+\lambda^2}. \end{aligned}\right\} \quad \ldots \quad \{9\}$$

Before we proceed to the application of these transformations, it will be necessary to premise some properties of the functions λ and λ'.

Let $\alpha = \cos.^{-1}(e^{-1})$ and $\alpha' = \cos.^{-1} e$.*

* This notation $\cos.^{-1} e$ must not be understood to signify $\frac{1}{\cos. e}$, but what is usually written thus, arc ($\cos. = e$). It is true that many authors use $\cos.^m$ A, Sin.^m A, &c. for $(\cos. \text{A})^m$, Sin. A^m; lest therefore the notation here adopted should appear capricious, it will not be irrelevant to explain its grounds. If φ be the characteristic mark of an operation performed on any symbol, x, $\varphi(x)$ may represent the result of that operation. Now to denote the repetition of the same operation, instead of $\varphi(\varphi(x))$; $\varphi(\varphi(\varphi(x)))$; &c. we may most elegantly write $\varphi^2(x)$; $\varphi^3(x)$; &c. Thus we use d^2x, $\Delta^3 x$, $\Sigma^2 x$, for ddx, $\Delta\Delta\Delta x$, $\Sigma\Sigma x$, &c. By the same analogy, since sin. x, cos. x, tan. x, log. x, &c. are merely *characteristic marks* to signify certain algebraic operations performed on the symbol x, (such as

$$\frac{\left\{1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.\right\}^{x\sqrt{-1}}-\left\{1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.\right\}^{-x\sqrt{-1}}}{2\sqrt{-1}};$$

&c.) we ought to write $\sin.^2 x$ for sin. sin. x, $\log.^3 x$ for log. log. log. x, and so on. To apply this to the inverse functions, we have $\varphi^n \varphi^m (x) = \varphi^{n+m}(x)$. Hence if $m = -n$ $\varphi^n \varphi^{-n}(x) = \varphi^0(x) = x$. with the operation ($\varphi$) performed *no times* on it, or merely x, that is, $\varphi^{-n}(x)$ must be such a quantity that its nth (φ) shall be x, or in other words $\varphi^{-n}(x)$ must represent the nth *inverse* function. It frequently happens that a peculiar characteristic symbol is appropriated to the inverse function. Let it be ψ, then $\varphi^{-n}, x = \psi^n, x$, and $\varphi^n x = \varphi^{2n}, \varphi^{-n}, x = \varphi^n(\varphi^n, \psi^n, x)$, hence $\psi^n, \varphi^n, x = x$, and therefore $\psi^{-n} x = \psi^{-n} \psi^n \varphi^n, x = \varphi^{+n}, x$. For instance d^{-n} $\text{V} = \int^n \text{V}$, $d^0\text{V} = \text{V}$. $-\Sigma^{-n}, x = \Delta^n x$. $a^n = 1$, with the operation of multiplying by a, n times performed on it, and $\therefore$ $a^{-n} = 1$, with the inverse operation so often performed on it, $= \frac{1}{a^n}$; $a^0 = 1$. Similarly $\sin.^{-1} x =$ arc ($\sin. = x$). $\cos.^{-2} x =$ arc ($\cos.^2 = x$) &c.—and if $c = 1 + \frac{1}{1} + \frac{1}{1.2} + \&c.$ $x = \log. c^x$ and $\therefore$ $c^x = \log.^{-1} x$. $c^{c^x} = \log.^{-2} x$, and $c^{c^{c\ldots(n)^x}} = \log.^{-n} x$, or the nth inverse logarithm of x. It is easy to carry on this idea, and its application to many very difficult operations in the higher branches will evince that it is somewhat more than a mere arbitrary contraction.

wherefore $\cos.\,\alpha = \frac{\lambda+\lambda^{-1}}{2}$ and $\cos.\,\alpha' = \frac{\lambda'+\lambda'^{-1}}{2}$.

Thus, $\lambda = c^{\alpha\sqrt{-1}}$ and $\lambda' = c^{\alpha'\sqrt{-1}}$, where $c = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} +$ &c. Hence, $\lambda^n + \lambda^{-n} = 2\,.\cos.\,n\alpha$, and $\lambda'^n + \lambda'^{-n} = 2\,.\cos.\,n\alpha'$; $\lambda^n - \lambda^{-n} = 2\sqrt{-1}\,.\,\text{cosin.}\,n\alpha$, $\lambda'^n - \lambda'^{-n} = 2\sqrt{-1}\,.\sin.\,n\alpha'$. Consequently, if k be any arc

$1 - 2\lambda^n\,.\cos.\,k + \lambda^{2n} = \lambda^n\left\{\lambda^n + \lambda^{-n} - 2\,.\cos.\,k\right\} = 4\lambda^n\,.\sin.\left(\frac{k+n\alpha}{2}\right).\sin.\left(\frac{k-n\alpha}{2}\right).$

In like manner

$$1 - 2\lambda'^n\,.\cos.\,k + \lambda'^{2n} = 4\lambda'^n\,.\sin.\left(\frac{k+n\alpha'}{2}\right).\sin.\left(\frac{k-n\alpha'}{2}\right).$$

We will now proceed to the application of these equations, and first, in Equation $\{1\}$ for θ substitute, successively, each of a series of angles

$$\theta_1;\ \theta_1 + \frac{2\pi}{n} = \theta_2;\ \theta_1 + \frac{4\pi}{n} = \theta_3;\ \ldots\ldots\ \theta_1 + \frac{2(n-1)\pi}{n} = \theta_n.$$

And let the resulting values of $r^{(1)}$ be

$$r^{(1)}_1;\ r^{(1)}_2;\ \ldots\ldots\ldots\ldots\ r^{(1)}_n$$

we have then

$$r^{(1)}_1\,.\,r^{(1)}_2\ldots\ldots r^{(1)}_n = a^n\,.\,\frac{(1-e^2)^n(1+\lambda^2)^n}{1-2\lambda^n\,.\cos.\,n\theta_1+\lambda^{2n}}\ldots\ldots\ldots\ldots\{1,1\}$$

for the product of the several denominators of the $r^{(1)}$ will be $\left\{1 - 2\lambda\,.\cos.\,\theta_1 + \lambda^2\right\}\left\{1 - 2\lambda\,.\cos.\,\theta_2 + \lambda^2\right\}\ldots\ldots\left\{1 - 2\lambda\,.\cos.\,\theta_n + \lambda^2\right\} = 1 - 2\lambda^n\,.\cos.\,n\theta_1 + \lambda^{2n}$ by Cotes's theorem.

This equation appears under an imaginary form when $e > 1$, but, since $\cos.^{-1} e$ is then a real angle, if we express it in α, it will then be free from imaginary symbols; thus

$$r_1^{(1)} \ldots\ldots r_n^{(1)} = \frac{a^n(1-e^2)^n(\lambda+\lambda^{-1})^n}{\lambda^n+\lambda^{-n}-2\,.\cos.\,n\theta_1} = \frac{\left\{2p\,.\cos.\,\alpha\right\}^n}{4\,.\sin.\,\frac{n(\theta_1+\alpha)}{2}\,.\sin.\,\frac{n(\theta_1-\alpha)}{2}} \quad \left\{1,2\right\}.$$

When $e=1$, or the conic section is a parabola, $\lambda=1$, and we find

$$r_1^{(1)} \ldots\ldots r_n^{(1)} = \frac{(2p)^n}{2\left\{1-\cos.\,n\theta_1\right\}} = (2p)^{n-1}\times\frac{p}{2}\,.\operatorname{cosec.}\left(\frac{n\theta_1}{2}\right)^2 \quad \left\{1,3\right\}.$$

A result of such remarkable simplicity, as deserves a more particular enunciation. Let then, in the diagram, fig. 1, S represent the focus of a parabola A,P,Q, and, having drawn any line SP_1, make n angles P_1SP_2, P_2SP_3 .... P_nSP_1, about S, all equal to each other; draw the axis ASM, and make the angle MSQ $= n$ times MSP_1; and if L represent the latus rectum, we shall have

$$SP_1\,.\,SP_2\,\ldots\ldots\,SP_n = L^{n-1}\,.\,SQ,$$

for, by the polar equation of the curve, $SQ = \frac{p}{2}\,.\operatorname{cosec.}\left(\frac{MSQ}{2}\right)^2$.

Thus, if SP_1 be coincident with SA, and n be odd, $\operatorname{cosec.}\frac{n\theta_1}{2} = 1$,

and $SP_1 \ldots\ldots SP_n = \frac{1}{4}L^n$. . . . . $\left\{1,4\right\}$;

but, if SP_1 be *perpendicular* to SA, and n still odd, $\operatorname{cosec.}\frac{n\theta_1}{2} = \sqrt{2}$. and $SP_1 \ldots\ldots SP_n = \frac{1}{2}L^n$ . . . . $\left\{1,5\right\}$.

If SP_1 be perpendicular to SA, but n of the form, $4m+2$

$SP_1 \ldots\ldots SP_n = \frac{1}{4}L^n$ . . . . $\left\{1,6\right\}$.

Lastly, if the angle $MSP_1 = \frac{\pi}{3}$, we shall find, provided n be of the form $6m + 1$,

$$SP_1 \ldots . SP_n = L^n \ldots\ldots \quad \{1,7\}.$$

Let us now resume the general equation $\{1,1\}$ and first, let $\theta_1 = o$, or, let one of the $r^{(1)}$ terminate in the second vertex, we have, for every value of n,

$$r^{(1)}_1 \ldots . r^{(1)}_n = p^n \cdot \frac{(1+\lambda^2)^n}{(1-\lambda^n)^2} = p^n \cdot \frac{(\lambda+\lambda^{-1})^n}{\left(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}\right)^2} = -\frac{(2p)^n}{4} \cdot \frac{(\cos.\alpha)^n}{\left(\sin.\frac{n\alpha}{2}\right)^2}.$$

$\{1,8\}$.

2. Let $\cos. n\theta_1 = -1$, and we obtain

$$r^{(1)}_1 \ldots . r^{(1)}_n = p^n \cdot \frac{(1+\lambda^2)^n}{(1+\lambda^n)^2} = p^n . = \frac{(\lambda+\lambda^{-1})^n}{\left(\lambda^{\frac{n}{2}} + \lambda^{-\frac{n}{2}}\right)^2} = \frac{(2p)^n}{4} \cdot \frac{(\cos.\alpha)^n}{\left(\cos.\frac{n\alpha}{2}\right)^2}.$$

$\{1,9\}$.

This embraces all the cases where n is of the form $(2m + 1)\frac{\pi}{\theta_1}$, and among the rest, when n is any odd number, and one of the $r^{(1)}$ terminates in the first vertex; when n is of the form $4m + 2$, and one of the $r^{(1)}$ perpendicular to the axis, &c.

3. Let $\cos. n\theta_1 = o$; then

$$r^{(1)}_1 \ldots . r^{(1)}_n = p^n \cdot \frac{(1+\lambda^2)^n}{1+\lambda^{2n}} = p^n \cdot \frac{(\lambda+\lambda^{-1})^n}{\lambda^n+\lambda^{-n}} = \frac{(2p)^n}{2} \cdot \frac{(\cos.\alpha)^n}{\cos. n\alpha} \quad \{1,10\}.$$

This includes the cases where n is of the form $(2m + 1)$. $\frac{\left(\frac{\pi}{2}\right)}{\theta_1}$, as for instance, where one of the $r^{(1)}$ is perpendicular to the axis and n odd, or, one inclined at an angle $\frac{\pi}{4}$, and n of the form $4m + 2$, or, lastly, where $\theta_1 = \frac{\pi}{6}$ and $n = 6m + 3$.

4. Let $\cos. n\theta_1 = \frac{1}{2}$, then

$$r_1^{(1)} \ldots\ldots r_n^{(1)} = p^n \cdot \frac{(1+\lambda^2)^n}{1-\lambda^n+\lambda^{2n}} = p^n \cdot \frac{(1+\lambda^2)^n \cdot (1+\lambda^n)}{1+\lambda^{3n}} \cdot \left\{1,11\right\}.$$

This takes place whenever n is of the form $(6m+1) \cdot \frac{\left(\frac{\pi}{3}\right)}{\theta_1}$, as when $\theta_1 = \frac{\pi}{3}$ and $n = (6m+1)$; $\theta_1 = \frac{\pi}{6}$, and $n = 12m + 2$, &c.

5. Let $\cos. n\theta_1 = -\frac{1}{2}$, then

$$r_1^{(1)} \ldots\ldots r_n^{(1)} = p^n \cdot \frac{(1+\lambda^2)^n}{1+\lambda^n+\lambda^{2n}} = p^n \cdot \frac{(1+\lambda^2)^n \cdot (1-\lambda^n)}{1-\lambda^{3n}} \quad \cdot \quad \cdot \left\{1,12\right\}.$$

Here n must be of the form $\frac{2\pi}{3\theta_1} \cdot (3m+1)$, and if $\theta_1 = \frac{2}{3}\pi$, n is of the form $3m + 1$.

We will now proceed to our second equation $\left\{2\right\}$, and by an operation exactly similar to that from which we obtained the equation $\left\{1,1\right\}$, we shall find

$$r_1^{(2)} \ldots\ldots r_n^{(2)} = a^n (1+\lambda^2)^n \cdot \frac{1-2e^n \cdot \cos. n\theta_1 + e^{2n}}{1-2\lambda^n \cdot \cos. n\theta_1 + \lambda^{2n}} \cdot \quad \cdot \quad \cdot \quad \cdot \left\{2,1\right\}$$

$$= a^n (\lambda + \lambda^{-1})^n \cdot \frac{1-2e^n \cdot \cos. n\theta_1 + e^{2n}}{\lambda^n + \lambda^{-n} - 2 \cdot \cos. n\theta_1} = \frac{a^n \left\{1-2e^n \cdot \cos. n\theta_1 + e^{2n}\right\}}{4 \cdot \sin. \frac{n(\theta_1+\alpha)}{2} \cdot \sin. \frac{n(\theta_1-\alpha)}{2}}.$$

This transformation in α, however, as it possesses no particular elegance in point of form, and much complexity, we shall henceforward omit, except in a few remarkable instances.

This equation, in the five particular cases just enumerated, gives

1. $n =$ any number, $\underset{1}{\theta} = o$.

$$\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}} = a^n (1 - e^n)^2 . \frac{(1+\lambda^2)^n}{(1-\lambda^n)^2} \quad . \quad . \quad . \quad . \quad \{2,2\}$$

2. $n = (2m+1) . \frac{\pi}{\underset{1}{\theta}}$.

$$\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}} = a^n . (1 + \lambda^2)^n . \left(\frac{1+e^n}{1+\lambda^n}\right)^2 \quad . \quad . \quad . \quad \{2,3\}$$

3. $n = (2m+1) . \frac{\pi}{2\underset{1}{\theta}}$.

$$\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}} = a^n . (1 + \lambda^2)^n . \frac{1+e^{2n}}{1+\lambda^{2n}} \quad . \quad . \quad . \quad \{2,4\}$$

4. $n = (6m+1) . \frac{\pi}{3\underset{1}{\theta}}$.

$$\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}} = a^n . (1 + \lambda^2)^n . \left(\frac{1+\lambda^n}{1+e^n}\right) . \left(\frac{1+e^{3n}}{1+\lambda^{3n}}\right) \quad . \quad . \quad \{2,5\}$$

5. $n = (6m+2) . \frac{\pi}{3\underset{1}{\theta}}$.

$$\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}} = a^n . (1 + \lambda^2)^n . \left(\frac{1-\lambda^n}{1-e^n}\right) . \left(\frac{1-e^{3n}}{1-\lambda^{3n}}\right) \quad . \quad . \quad \{2,6\}$$

Our third equation $\{3\}$, gives immediately

$$\frac{\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(n)}}}{\underset{1}{r^{(1)}} \ldots\ldots \underset{n}{r^{(1)}}} = \frac{1 - 2e^n . \cos. n\underset{1}{\theta} + e^{2n}}{(1-e^2)^n} \quad . \quad . \quad . \quad . \quad \{3,1\}$$

which, in the abovementioned cases, becomes

1. $$\frac{\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}}}{\underset{1}{r^{(1)}} \ldots\ldots \underset{n}{r^{(1)}}} = \frac{(1-e^n)^2}{(1-e^2)^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \{3,2\}$$

2. $$\frac{\underset{1}{r^{(2)}} \ldots\ldots \underset{n}{r^{(2)}}}{\underset{1}{r^{(1)}} \ldots\ldots \underset{n}{r^{(1)}}} = \frac{(1+e^n)^2}{(1-e^2)^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \{3,3\}$$

$$3.\ \frac{r^{(2)}_{1} \ldots\ldots r^{(2)}_{n}}{r^{(1)}_{1} \ldots\ldots r^{(1)}_{n}} = \frac{1+e^{2n}}{(1-e^{2})^{n}} \quad \ldots\ldots\ldots \quad \{3,4\}$$

$$4.\ \frac{r^{(2)}_{1} \ldots\ldots r^{(2)}_{n}}{r^{(1)}_{1} \ldots\ldots r^{(1)}_{n}} = \frac{1+e^{3n}}{(1+e^{n})(1-e^{2})^{n}} \quad \ldots\ldots\ldots \quad \{3,5\}$$

$$5.\ \frac{r^{(2)}_{1} \ldots\ldots r^{(2)}_{n}}{r^{(1)}_{1} \ldots\ldots r^{(1)}_{n}} = \frac{1-e^{3n}}{(1-e^{n})(1-e^{2})^{n}} \quad \ldots\ldots\ldots \quad \{3,6\}.$$

Let us now, instead of taking equal angles round the first focus, take a series of eccentric anomalies, in arithmetical progression

$$\varpi_{1};\ \varpi_{2} = \varpi_{1} + \frac{2\pi}{n};\ \ldots\ldots \varpi_{n} = \varpi_{1} + \frac{2(n-1)\pi}{n},$$

and let the resulting values of $r^{(1)}$ be, as before

$$r^{(1)}_{1};\ r^{(1)}_{2};\ \ldots\ldots r^{(1)}_{n}.$$

we get then, by the same process,

$$r^{(1)}_{1} \ldots\ldots r^{(1)}_{n} = a^{n} . \frac{1 - 2\lambda^{n} . \cos . n\varpi_{1} + \lambda^{2n}}{(1+\lambda^{2})^{n}} \quad \ldots \quad \{4,1\}$$

This gives, in the five cases; 1st, when n is any number, and $\varpi_{1} = o$, (or one of the $r^{(1)}$ terminates in the first vertex); 2dly, when $n = (2m+1) . \frac{\pi}{\varpi_{1}}$; 3dly, $n = (2m+1) . \frac{\pi}{2\varpi_{1}}$; 4thly, $n = (6m+1) . \frac{\pi}{3\varpi_{1}}$; and 5thly, $n = (6m+2) . \frac{\pi}{3\varpi_{1}}$; the following equations

$$1.\ r^{(1)}_{1} \ldots\ldots r^{(1)}_{n} = a^{n} . \frac{\left(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}\right)^{2}}{(\lambda+\lambda^{-1})^{n}} \quad \ldots\ldots\ldots \quad \{4,2\}$$

$$2.\ r^{(1)}_{1} \ldots\ldots r^{(1)}_{n} = a^{n} . \frac{\left(\lambda^{\frac{n}{2}} + \lambda^{-\frac{n}{2}}\right)^{2}}{\lambda+\lambda^{-1})^{n}} \quad \ldots\ldots\ldots \quad \{4,3\}$$

$$3.\ r^{(1)}_{1} \ldots\ldots r^{(1)}_{n} = a^n \cdot \frac{\lambda^n + \lambda^{-n}}{(\lambda + \lambda^{-1})^n} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \left\{4,4\right\}$$

$$4.\ r^{(1)}_{1} \ldots\ldots r^{(1)}_{n} = a^n \cdot \frac{\left(\lambda^{\frac{3n}{2}} + \lambda^{-\frac{3n}{2}}\right)}{(\lambda + \lambda^{-1}) \cdot \left(\lambda^{\frac{n}{2}} + \lambda^{-\frac{n}{2}}\right)} \cdot \cdot \cdot \cdot \left\{4,5\right\}$$

$$5.\ r^{(1)}_{1} \ldots\ldots r^{(1)}_{n} = a^n \cdot \frac{\left(\lambda^{\frac{3n}{2}} - \lambda^{-\frac{3n}{2}}\right)}{(\lambda + \lambda^{-1}) \cdot \left(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}\right)} \cdot \cdot \cdot \cdot \left\{4,6\right\}.$$

The Equation $\{5\}$ may be treated in the same manner, for the values of ϖ, being ϖ_1; $\varpi_1 + \frac{2\pi}{n}$; ...... $\varpi_1 + \frac{2(n-1)\pi}{n}$, those of $\pi - \varpi$ in an inverted order will be (if $\nu = -\left\{\varpi_1 + \frac{(n-2)\pi}{n}\right\}$)

$$\nu,\ \nu + \frac{2\pi}{n}, \ldots\ldots\ldots \nu + \frac{2(n-1)\pi}{n}.$$

Hence we find

$$r^{(2)}_{1} \ldots\ldots r^{(2)}_{n} = a^n \cdot \frac{1 - 2\lambda^n \cdot \cos.\ n\nu + \lambda^{2n}}{(1+\lambda^2)^n},$$

or, since $\cos.\ \nu = \cos.\ n\ (\pi + \varpi_1)$

$$r^{(2)}_{1} \ldots\ldots r^{(2)}_{n} = a^n \cdot \frac{1 - 2\lambda^n \cdot \cos.\ n\ (\pi + \varpi_1) + \lambda^{2n}}{(1+\lambda^2)^n} \cdot \cdot \cdot \left\{5,1\right\}.$$

As this case, however, is manifestly similar to that of $\{4,1\}$, we shall pursue it no farther.

The 6th of the equations, in page 9, offers, however, some results worthy of consideration. By treating it like the rest, it becomes

$$\frac{r^{(1)}_{1} \ldots\ldots r^{(1)}_{n}}{r^{(2)}_{1} \ldots\ldots r^{(2)}_{n}} = \frac{1 - 2\lambda^n \cdot \cos.\ n\varpi_1 + \lambda^{2n}}{1 - 2\lambda^n \cdot \cos.\ n\ (\pi + \varpi_1) + \lambda^{2n}} \cdot \cdot \cdot \cdot \left\{6,1\right\}.$$

1. When $\varpi_1 = o$, it becomes

$$\frac{(1-\lambda^n)^2}{1-2\lambda^n \,.\, \cos.\,(n\pi)+\lambda^{2n}}$$

If then, n be even, this is equal to unity, as it evidently ought, but if odd

$$\frac{\underset{1}{r^{(1)}}\ldots\ldots\underset{n}{r^{(1)}}}{\underset{1}{r^{(2)}}\ldots\ldots\underset{n}{r^{(2)}}}=\left(\frac{1-\lambda^n}{1+\lambda^n}\right)^2=-\left(\tan.\frac{n\alpha}{2}\right)^2 \quad . \quad . \quad . \quad . \quad \{6,2\}.$$

2. Let $\underset{1}{\varpi}=\frac{\pi}{3}$, and $n=6m+1$, and we find

$$\frac{\underset{1}{r^{(1)}}\ldots\ldots\underset{n}{r^{(1)}}}{\underset{1}{r^{(2)}}\ldots\ldots\underset{n}{r^{(2)}}}=\left(\frac{1+\lambda^{3n}}{1-\lambda^{3n}}\right)\,.\,\left(\frac{1-\lambda^n}{1+\lambda^n}\right)=\tan.\frac{n\alpha}{2}\,.\,\text{cotan.}\frac{3na}{2} \quad \{6,3\}.$$

We come now to our 7th Equation, which will afford us results, more complicated indeed, yet equally interesting. By applying the same method of transformation to it, we shall find, (supposing $\underset{1}{\phi}, \underset{2}{\phi}, \ldots\ldots \underset{n}{\phi}=\underset{1}{\phi}+\frac{2(n-1)\pi}{n}$, to be written for ϕ, and $\underset{1}{R}, \underset{2}{R}, \ldots\ldots \underset{n}{R}$ to denote the resulting values of R)

$$\underset{1}{R}\ldots\ldots\underset{n}{R}=a^n\,.\,\frac{(1+\lambda^2)^n\,(1-e^2)^{\frac{n}{2}}}{\left\{1-2\lambda^n\,.\,\cos.\,n\underset{1}{\phi}+\lambda^{2n}\right\}^{\frac{1}{2}}\,.\,\left\{1-2\lambda^n\,.\,\cos.\,n\,(\pi+\underset{1}{\phi})+\lambda^{2n}\right\}^{\frac{1}{2}}}.$$

$\{7,1\}$.

1. If n be even, $\cos.\,n\underset{1}{\phi}=\cos.\,n\,(\pi+\underset{1}{\phi})$ and, since $1-e^2$ $=\left\{\frac{1-\lambda^2}{1+\lambda^2}\right\}^2$, this becomes

$$\underset{1}{R}\ldots\ldots\underset{n}{R}=a^n\cdot\frac{(1-\lambda^2)^n}{1-2\lambda^n\,.\,\cos.\,n\underset{1}{\phi}+\lambda^{2n}} \quad . \quad . \quad . \quad . \{7,2\}.$$

2. If n be odd, $\cos.\,n\underset{1}{\phi}=-\cos.\,n\,(\pi+\underset{1}{\phi})$, whence

$$\underset{1}{R}\ldots\ldots\underset{n}{R}=a^n\,.\,\frac{(1-\lambda^2)^n}{\left\{1-2\lambda^n\,.\,\cos.\,n\underset{1}{\phi}+\lambda^{2n}\right\}^{\frac{1}{2}}\,.\,\left\{1+2\lambda^n\,.\,\cos.\,n\underset{1}{\phi}+\lambda^{2n}\right\}^{\frac{1}{2}}}$$

$$= a^n \cdot \frac{(1-\lambda^2)^n}{\left\{1-2\lambda^{2n}\,(2\,.\cos.\,n\underset{1}{\phi}^2-1)+\lambda^{4n}\right\}^{\frac{1}{2}}}, \text{ or}$$

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{\left\{1-2\lambda^{2n}\,.\cos.\,2n\underset{1}{\phi}+\lambda^{4n}\right\}^{\frac{1}{2}}} \quad \ldots\ldots \quad \{7,3\}.$$

Let $\underset{1}{\phi} = o$, or, let the extremity of one of the R lie in the principal vertex, If n be even

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{(1-\lambda^n)^2} = a^n \cdot \frac{(\sin.\,\alpha)^n}{(\sin.\,\frac{n}{2}\,\alpha)^2} \quad \ldots\ldots \quad \{7,4\}.$$

If odd,

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{1-\lambda^{2n}} = a^n \cdot \frac{(\sin.\,\alpha)^n}{\sin.\,n\alpha} \quad \ldots\ldots \quad \{7,5\}.$$

Let $\underset{1}{\phi} = \frac{\pi}{2}$, and $n = 4m + 2$, In this case, two out of the R are at right angles to the axis, and

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{(1+\lambda^n)^2} \quad \ldots\ldots \quad \{7,6\}.$$

Again, let *one only* of the R be perpendicular to the axis, and

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{1+\lambda^{2n}} \quad \ldots\ldots \quad \{7,7\}.$$

Here n is of course odd.

Next, let one of the R be inclined at an angle $\frac{\pi}{4}$, to the axis. If $n = 4m + 2$,

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{1+\lambda^{2n}} \quad \ldots\ldots \quad \{7,8\},$$

and it is curious to observe, that this expression is the same function of a, λ, n, as that of $\{7,7\}$.

If n be of the form $2m + 1$,

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{\sqrt{1+\lambda^{4n}}} \quad \ldots\ldots \quad \{7,9\}.$$

Lastly, let n be of the form $6m + 2$, and $\underset{1}{\phi} = \frac{\pi}{3}$, then

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n (1-\lambda^n)}{(1-\lambda^{3n})} \quad \ldots\ldots\ldots \quad \{7,10\};$$

but, if $n = 6m + 1$,

$$\underset{1}{R}\ldots\ldots\underset{n}{R} = a^n \cdot \frac{(1-\lambda^2)^n}{\sqrt{(1+\lambda^{2n}+\lambda^{4n})}} \quad \ldots\ldots\ldots \quad \{7,11\}.$$

These are always imaginary expressions when $e > 1$ and n odd. In fact, R, in the hyperbola, must be written

$$\frac{a\sqrt{e^2-1}}{\sqrt{e^2 \cdot \cos. \varphi^2 - 1}} \text{ instead of } \frac{a\sqrt{1-e^2}}{\sqrt{1-e^2 \cdot \cos. \varphi^2}}$$

now $\sqrt{e^2-1} = e \cdot \frac{1-\lambda'^2}{1+\lambda'^2}$. Thus, this expression, like the rest, is easily transformed, in functions of a, λ, λ', the λ' is now real, and the part involving λ will always be of the form $f(\lambda^m \pm \lambda^{-m})$, and therefore readily expressed in trigonometrical functions.

Before we proceed farther, it will be necessary to premise a transformation of COTES's formula, which we shall have occasion to make use of. It is as follows:

$$\left.\begin{array}{l} \sin. (A+B) \cdot \sin. (A-B) = 2^{2n-2} \cdot P \cdot Q, \text{ where} \\ P = \sin. \left(\frac{A+B}{n}\right) \cdot \sin. \left(\frac{\pi+(A+B)}{n}\right) \cdot \sin. \left(\frac{2\pi+(A+B)}{n}\right) \ldots\ldots \\ \sin. \left(\frac{(n-1)\pi+(A+B)}{n}\right) \\ Q = \sin. \left(\frac{A-B}{n}\right) \cdot \sin. \left(\frac{\pi+(A-B)}{n}\right) \cdot \sin. \left(\frac{2\pi+(A-B)}{n}\right) \ldots\ldots \\ \sin. \left(\frac{(n-1)\pi+(A-B)}{n}\right) \end{array}\right\} (a)$$

The demonstration is extremely simple,

$$1 - 2x^n \cdot \cos. a + x^{2n} = \left(1 - 2x \cdot \cos. \frac{a}{n} + x^2\right)\left(1 - 2x \cdot \cos. \frac{a+2\pi}{n} + x^2\right) \ldots\ldots\ldots \left(1 - 2x \cdot \cos. \frac{a+2(n-1)\pi}{n} + x^2\right),$$

or dividing by x^n

$$x^n + x^{-n} - 2 \cdot \cos. a = \left(x + x^{-1} - 2 \cdot \cos. \frac{a}{n}\right) \ldots\ldots\ldots$$

$\left(x + x^{-1} - 2 \,.\, \cos. \frac{a+2(n-1)\pi}{n}\right)$.

Let $x + x^{-1} = 2 \,.\, \cos. c$; then $x^n + x^{-n} = 2 \,.\, \cos. nc$, and $\cos. nc - \cos. a = 2^{n-1} \left(\cos. c - \cos. \frac{a}{n}\right) \ldots\ldots \left(\cos. c - \cos. \frac{a+2(n-1)\pi}{n}\right)$, that is (by the formula $\cos. x - \cos. y = -2 \,.\, \sin. \frac{x+y}{2} \,.\, \sin. \frac{x-y}{2}$) . $\sin. \left(\frac{a+nc}{2}\right) . \sin. \left(\frac{a-nc}{2}\right) = 2^{2n-2}$

$$. \sin. \tfrac{1}{2} \left(\frac{a}{n} + c\right) . \sin. \tfrac{1}{2} \left(\frac{a+2\pi}{n} + c\right) . \&c.$$

$$. \sin. \tfrac{1}{2} \left(\frac{a}{n} - c\right) . \sin. \tfrac{1}{2} \left(\frac{a+2\pi}{n} - c\right) . \&c.$$

Let $\frac{a+nc}{2} = A + B$, and $\frac{a-nc}{2} = A - B$, and by substitution, the formula under consideration results.

This immediately gives the following

$$\left.\begin{array}{l} \cos. (A + B) \,.\, \cos. (A - B) = 2^{2n-2} \,.\, P \,.\, Q, \text{ where} \\ P = \sin. \left\{\frac{\left(\frac{\pi}{2}\right) - (A+B)}{n}\right\} . \sin. \left\{\frac{3\left(\frac{\pi}{2}\right) - (A+B)}{n}\right\} \ldots\ldots \\ \sin. \left\{\frac{(2n-1) . \left(\frac{\pi}{2}\right) - (A+B)}{n}\right\} \\ Q = \sin. \left\{\frac{\left(\frac{\pi}{2}\right) - (A-B)}{n}\right\} . \sin. \left\{\frac{3\left(\frac{\pi}{2}\right) - (A-B)}{n}\right\} \ldots\ldots \\ \sin. \left\{\frac{(2n-1)\left(\frac{\pi}{2}\right) - (A-B)}{n}\right\} \end{array}\right\} (b)$$

and also,

$$\left.\begin{array}{l} \cos. (A + B) \,.\, \sin. (A - B) = 2^{2n-2} \,.\, P \,.\, Q, \text{ where} \\ P = \sin. \left\{\frac{\left(\frac{\pi}{2}\right) - (A+B)}{n}\right\} . \sin. \left\{\frac{3\left(\frac{\pi}{2}\right) - (A+B)}{n}\right\} \ldots\ldots \\ \sin. \left\{\frac{(2n-1)\left(\frac{\pi}{2}\right) - (A+B)}{n}\right\} \\ Q = \sin. \left(\frac{A-B}{n}\right) . \sin. \left(\frac{\pi + (A-B)}{n}\right) \ldots . \sin. \left(\frac{(n-1)\pi + (A-B)}{n}\right). \end{array}\right\} (c)$$

Let B $= o$, and (a) gives

$$\sin. A = 2^{n-1} . \sin. \frac{A}{n} . \sin. \frac{\pi + A}{n} . \sin. \frac{2\pi + A}{n} \dots \sin. \frac{(n-1)\pi + A}{n}. \quad (d)$$

If $A = \frac{\pi}{2}$, this becomes

$$1 = 2^{n-1} . \sin. \frac{\pi}{2n} . \sin. \frac{3\pi}{2n} . \sin. \frac{5\pi}{2n} \dots\dots \sin. \frac{(2n-1)\pi}{2n}. \quad (e)$$

(b) gives by making B $= o$,

$$\cos. A = 2^{n-1} . \sin. \frac{\left(\frac{\pi}{2}\right) - A}{n} . \sin. \frac{3\left(\frac{\pi}{2}\right) - A}{n} \dots \sin. \frac{(2n-1)\left(\frac{\pi}{2}\right) - A}{n}. \quad (f)$$

If $A = \frac{\pi}{3}$, this gives

$$1 = 2^{n} . \sin. \frac{\pi}{6n} . \sin. \frac{7\pi}{6n} . \sin. \frac{13\pi}{6n} . \sin. \frac{19\pi}{6n} \dots\dots \sin. \frac{(6n-5)\pi}{6n}. \quad (g)$$

Equation (c) divided by (b) gives, (putting A, for A — B).

$$\tan. A = \frac{\sin. \frac{A}{n} . \sin. \frac{\pi + A}{n} . \sin. \frac{2\pi + A}{n} \dots\dots \sin. \frac{(n-1)\pi + A}{n}}{\sin. \left\{ \frac{\left(\frac{\pi}{2}\right) - A}{n} \right\} . \sin. \left\{ \frac{3\left(\frac{\pi}{2}\right) - A}{n} \right\} \dots \sin. \left\{ \frac{(2n-1)\left(\frac{\pi}{2}\right) - A}{n} \right\}}. \quad (h)$$

But, to return from this digression, let us take Equation $\{8\}$, and putting it into this form

$$\rho = \frac{2p . (1+\lambda^2)^2 . \sin. \left(\frac{\pi}{2} - \psi\right)}{(1 - 2\lambda . \cos. \psi + \lambda^2)(1 - 2\lambda . \cos. (\pi - \psi) + \lambda^2)}$$

for ψ substitute each of a series of angles, n in number

$$\psi_1 ; \ \psi_1 + \frac{\pi}{n} ; \ \psi_1 + \frac{2\pi}{n} ; \dots\dots \psi_1 + \frac{(n-1)\pi}{n},$$

and let $\rho_1, \rho_2, \dots . \rho_n$ be the resulting values of ρ.

The values of $\sin. \left(\frac{\pi}{2} - \psi\right)$ in an inverted order, are (if $\nu = - \left\{ \psi_1 + \frac{n-2}{n} . \left(\frac{\pi}{2}\right) \right\}$) $\sin. \nu$; $\sin. \left(\nu + \frac{\pi}{n}\right)$; $\sin. \left(\nu + \frac{2\pi}{n}\right)$; .... $\sin. \left(\nu + \frac{n-1}{n}\pi\right)$, and their product, $= \sin. \frac{n\nu}{n} . \sin. \frac{\pi + n\nu}{n}$ .... $\sin. \frac{(n-1)\pi + n\nu}{n} = \frac{\sin. n\nu}{2^{n-1}}$ (by Equation (d)) $= \frac{\sin. n\left(\frac{\pi}{2} + \psi_1\right)}{2^{n-1}}$.

Again, the values of ψ being $\underset{1}{\psi}, \ldots . \underset{1}{\psi} + \frac{(n-1)\pi}{n}$, those of $\pi + \psi$, or $2\pi - (\pi - \psi)$ will be

$$\underset{1}{\psi} + \frac{n\pi}{n}, \underset{1}{\psi} + \frac{(n+1)\pi}{n}; \ldots . \underset{1}{\psi} + \frac{(2n-1)\pi}{n}.$$

Now, cos. $2\pi - (\pi - \psi) =$ cos. $(\pi - \psi)$. Hence, the product of all the denominators of $\underset{1}{\rho}, \underset{2}{\rho}$, &c. will be $(1 - 2\lambda \,.\, \text{cos.}\, \underset{1}{\psi} + \lambda^2)(1 - 2\lambda \,.\, \text{cos.}\, (\underset{1}{\psi} + \frac{\pi}{n}) + \lambda^2) \ldots (1 - 2\lambda \,.\, \text{cos.}\, (\underset{1}{\psi} + \frac{(n-1)\pi}{n}) + \lambda^2)$. cos. $(\underset{1}{\psi} + \frac{\pi}{n}) + \lambda^2) \ldots (1 - 2\lambda \,.\, \text{cos.}\, (\underset{1}{\psi} + \frac{(2n-1)\pi}{n}) + \lambda^2) =$, by Cotes's formula

$$1 - 2\lambda^{2n} \,.\, \text{cos.}\, 2n\,\underset{1}{\psi} + \lambda^{4n}.$$

Thus we have, combining these separate processes,

$$\left.\begin{aligned} \underset{1}{\rho} \,.\, \underset{2}{\rho} \ldots . \underset{n}{\rho} &= 2a^n (1 - e^2)^n \,.\, (1 + \lambda^2)^{2n} \,.\, \frac{\text{sin.}\, n \left(\frac{\pi}{2} + \underset{1}{\psi}\right)}{1 - 2\lambda^{2n} \,.\, \text{cos.}\, 2n\underset{1}{\psi} + \lambda^{4n}} \\ &= 2p^n \,.\, \frac{(1+\lambda^2)^{2n} \,.\, \text{sin.}\, n \left(\frac{\pi}{2} + \underset{1}{\psi}\right)}{1 - 2\lambda^{2n} \,.\, \text{cos.}\, 2n\underset{1}{\psi} + \lambda^{4n}} = 2a^n \,.\, \frac{(1-\lambda^2)^{2n} \,.\, \text{sin.}\, n \left(\frac{\pi}{2} + \underset{1}{\psi}\right)}{1 - 2\lambda^{n} \,.\, \text{cos.}\, 2n\underset{1}{\psi} + \lambda^{4n}} \end{aligned}\right\} \{ \quad \}$$

1. Let n be any number of the form $4m + 1$; then

$$\underset{1}{\rho} \ldots . \underset{n}{\rho} = 2p^n \,.\, \frac{(1+\lambda^2)^{2n} \,.\, \text{cos.}\, n\underset{1}{\psi}}{1 - 2\lambda^{2n} \,.\, \text{cos.}\, 2n\underset{1}{\psi} + \lambda^{4n}} \quad . \quad . \quad . \quad . \quad . \quad \{8,2\}.$$

If now $\underset{1}{\psi} = o$, or one of the ρ coincide with the transverse axis,

$$\underset{1}{\rho} \ldots . \underset{n}{\rho} = \frac{2p^n (1+\lambda^2)^{2n}}{(1-\lambda^{2n})^2} = 2a^n \,.\, \frac{(1-\lambda^2)^{2n}}{(1-\lambda^{2n})^2} \quad . \quad . \quad . \quad . \quad . \quad \{8,3\}.$$

Let, next, $\underset{1}{\psi} = \frac{\pi}{4}$ $\{n$ continuing $= 4m + 1\}$ and,

$$\underset{1}{\rho} \ldots . \underset{n}{\rho} = \pm p^n \sqrt{2} \,.\, \frac{(1+\lambda^2)^{2n}}{1+\lambda^{4n}} \quad . \quad . \quad . \quad . \quad \{8,4\}.$$

In the parabola, $\lambda = 1$. Hence in this case,

$$\underset{1}{\rho} \cdots \underset{n}{\rho} = \pm (2L)^n \cdot \sqrt{\tfrac{1}{2}} \quad . \quad . \quad . \quad . \quad \{8,5\}.$$

In fig. 2, let ASM be the axis of a parabola, BAC a tangent at the vertex, S the focus—bisect the angle BAM by $\underset{1}{AP}$, and draw $4m$ other lines $\underset{2}{AP}$, $\underset{3}{AP}$, ... $\underset{n}{AP}$, so that if $\underset{1}{PA}$, $\underset{2}{PA}$, &c. be produced through A, this system of lines shall make equal angles around A, then, neglecting the sign

$$\underset{1}{AP} \cdot \underset{2}{AP} \cdots \underset{n}{AP} = (2L)^n \cdot \sqrt{\tfrac{1}{2}}.$$

If $n = 4m + 3$, the resulting value of $\underset{1}{\rho} \ldots \underset{n}{\rho}$ is of the same form as $\{8,2\}$ with the exception of a different sign.

If n be of the form $4m$, or $4m + 2$,

$$\underset{1}{\rho} \cdots \underset{n}{\rho} = \pm \frac{2a^n (1-\lambda^2)^{2n} \cdot \sin. n\underset{1}{\psi}}{1 - 2\lambda^{2n} \cdot \cos. 2n\underset{1}{\psi} + \lambda^{4n}} \quad . \quad . \quad . \quad . \quad . \quad \{8,6\},$$

where the sign $+$, or $-$, is to be used, according as n is of the former, or latter form.

When $n = 4m + 2$, if $\underset{1}{\psi} = \frac{\pi}{4}$, this becomes

$$\underset{1}{\rho} \cdots \underset{n}{\rho} = \pm 2p^n \cdot \frac{(1+\lambda^2)^{2n}}{(1+\lambda^{2n})^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \{8,7\},$$

and when $\lambda = 1$.

$$\underset{1}{\rho} \cdots \underset{n}{\rho} = \pm \frac{(2L)^n}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \{8,8\}.$$

Thus, in the last mentioned construction, if the number of lines be $4m + 2$, instead of $4m + 1$

$$\underset{1}{AP} \cdot \underset{2}{AP} \cdots \underset{n}{AP} = \frac{(2L)^n}{2}.$$

The ninth of our primitive equations gives, if the angles

ϖ_1; $\varpi_2 = \varpi_1 + \frac{2\pi}{n}$; .... $\varpi_n = \varpi_1 + \frac{2(n-1)\pi}{n}$, be written for ϖ, and for θ $\Big\{$a function of ϖ, given by the equation

$$\cot.\ \tfrac{1}{2}\theta = \frac{1+\lambda}{1-\lambda}\ .\ \tan.\ \tfrac{1}{2}\varpi.\Big\}$$

be put θ_1; .... θ_n, the results of that substitution

$$\cos.\ \theta_1 . \cos.\ \theta_2 \ldots \cos.\ \theta_n = e^{-n} . \left(\frac{\lambda}{\lambda'}\right)^n . \frac{1 - 2'\lambda^n . \cos.\ n\varpi_1 + \lambda'^{2n}}{1 - 2\lambda^n . \cos.\ n\varpi_1 + \lambda^{2n}} . \left\{9,1\right\}.$$

If $\varpi_1 = o$, whatever be n,

$$\cos.\theta_1 \ldots \cos.\theta_n = e^{-n} . \left(\frac{\lambda}{\lambda'}\right)^n . \left(\frac{1-\lambda'^n}{1-\lambda}\right)^2 = e^{-n} . \left\{\frac{\lambda'^{\frac{n}{2}} - \lambda'^{-\frac{n}{2}}}{\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}}\right\}^2 . \left\{9,2\right\}.$$

If $n = (2m+1)\ \frac{\pi}{\varpi_1}$

$$\cos.\theta \ldots . \cos.\theta_n = e^{-n} . \left\{\frac{\lambda'^{\frac{n}{2}} + \lambda'^{-\frac{n}{2}}}{\lambda^{\frac{n}{2}} + \lambda^{-\frac{n}{2}}}\right\}^2 \quad \ldots\ldots \quad \left\{9,3\right\}.$$

If $n = (2m+1) . \frac{\left(\frac{\pi}{2}\right)}{\varpi_1}$,

$$\cos.\ \theta_1 \ldots . \cos.\ \theta_n = e^{-n} . \frac{\lambda'^n + \lambda'^{-n}}{\lambda^n + \lambda^{-n}} \quad \ldots\ldots\ldots \quad \left\{9,4\right\}.$$

If $n = (6m+1) . \frac{\left(\frac{\pi}{3}\right)}{\varpi_1}$;

$$\cos\ \theta_1 \ldots . \cos.\ \theta_n = e^{-n} . \left(\frac{\lambda^{\frac{n}{2}} + \lambda^{-\frac{n}{2}}}{\lambda'^{\frac{n}{2}} + \lambda'^{-\frac{n}{2}}}\right) . \left(\frac{\lambda'^{\frac{3n}{2}} + \lambda'^{-\frac{3n}{2}}}{\lambda^{\frac{3n}{2}} + \lambda^{-\frac{3n}{2}}}\right) \left\{9,5\right\}.$$

If $n = (6m+2) . \frac{\left(\frac{\pi}{3}\right)}{\varpi_1}$,

$$\cos.\ \theta_1 \ldots . \cos.\ \theta_n = e^{-n} . \left(\frac{\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}}{\lambda'^{\frac{n}{2}} - \lambda'^{-\frac{n}{2}}}\right) . \left(\frac{\lambda'^{\frac{3n}{2}} - \lambda'^{-\frac{3n}{2}}}{\lambda^{\frac{3n}{2}} - \lambda^{-\frac{3n}{2}}}\right) \left\{9,6\right\}.$$

These theorems, however simple their algebraic expressions, it must immediately be seen, become for the most part complicated and unintelligible when geometrically enunciated. They are indeed (if we may in any case be allowed to consider a curve as unidentified with its equation) properties rather of the equations of the conic sections, than of the curves themselves,—of a limited number of disjoined points determined according to a certain law, rather than a series of consecutive ones composing a line.

JOHN F. W. HERSCHEL.

Slough, Oct. 6, 1812.

Fig. 1.

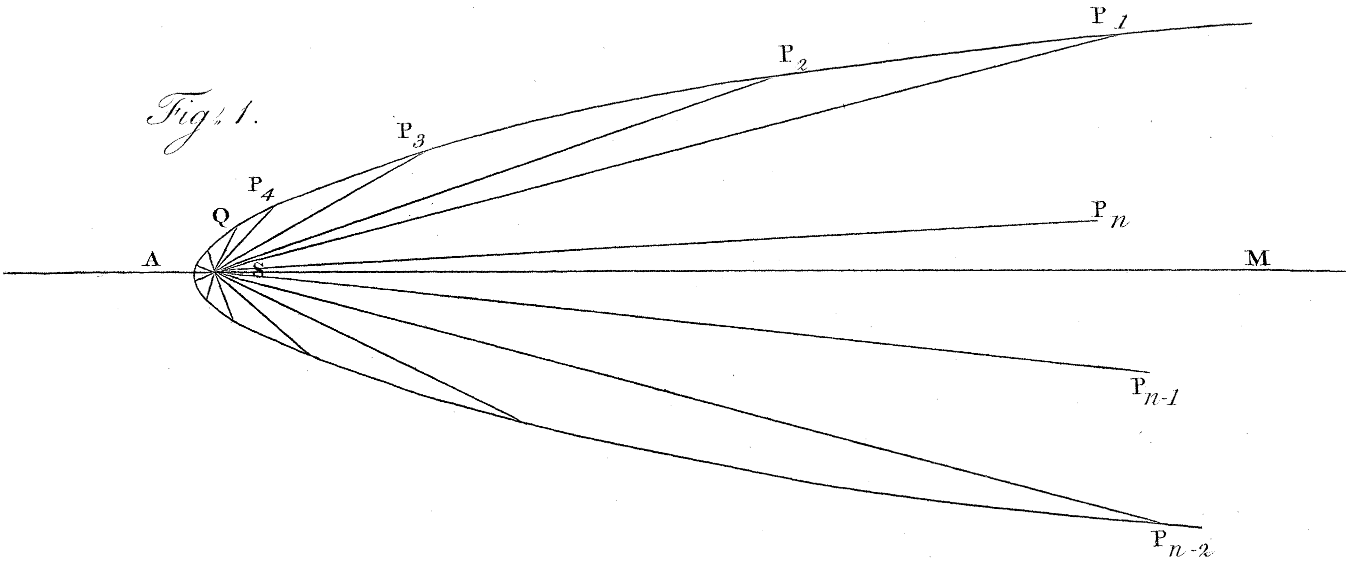
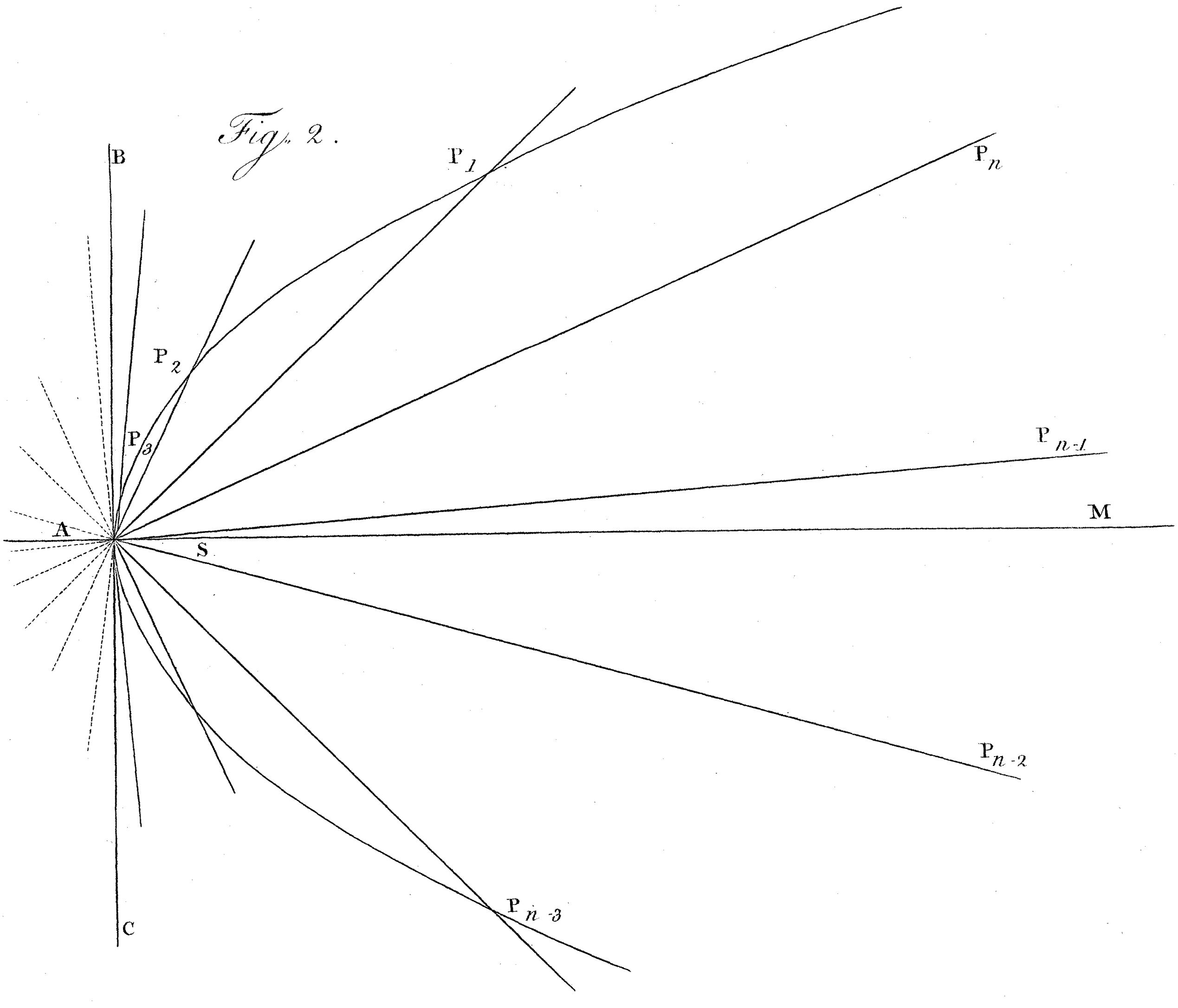


Fig. 2.



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